Spinor representations of the Virasoro and super-Virasoro algebras for conformal spin to be equal $\frac{1}{k}$

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Abstract

It is considered here the possibility of unitary spinor representations of the Virasoro and super-Virasoro algebras for conformal spin to be equal $\frac{1}{k}$; k are integers.

Virasoro group on the two-dimensional plane of complex variable z is defined by the following group of conformal transformations :

$$z'^n = \frac{az^n + b}{cz^n + d}; \qquad ad - bc = 1; n \in Z \tag{1}$$

Virasoro generators L_n are determined by transformations:

$$L_n = z^{-n+1}\frac{d}{dz} \tag{2}$$

They satisfy the usual classical commutator algebra:

$$[L_n, L_m] = (n - m)L_{n+m} \tag{3}$$

Quantum mechanical applications of the Virasoro algebra lead to central extensions:

$$[L_n, L_m] = (n-m)L_{n+m} + c_{mn} \quad (4)$$

Where

$$c_{mn} = \frac{c}{12} m(m^2-1)\delta_{m,-n} \quad (5)$$

Unitary conditions for these operators L_n are

$$L_n^\dagger = L_{-n} \quad (6)$$

In physical applications (vertices of string theory) we use operator functions F(z):

$$F(z) = \sum_n F_n z^n \quad (7)$$

which are representations of this Virasoro group corresponding to conformal spin j:

$$[L_n, F(z)] = (z^{-n+1}\frac{d}{dz} - jnz^{-n})F(z) \quad (8)$$

Then we should have for any conformal spin j :

$$[L_n, F_r] = ((j-1)n - r)F_{n+r} \quad (9)$$

Unitary representations of the Virasoro and super-Virasoro algebras are found for conformal spins j to be equal $0, 1$ and $\frac{1}{2}$ [1].

For j=1 we have:

$$F_{j=1}(z) = \sum_{n=1} a_n z^n + p + \sum_{n=1} a_{-n} z^{-n} \quad (10)$$

$$[a_n, a_m] = n\delta_{n,-m}; \quad n > 0 \quad a_n^\dagger = a_{-n}; \quad (11)$$

$$L_n = \frac{1}{2}\sum_{m\neq 0} a_{n-m}a_m + pa_n; \quad n \neq 0; \quad L_0 = \sum_{m\neq 0} a_{-m}a_m + \frac{1}{2}p^2; \quad L_n^\dagger = L_{-n} \quad (12)$$

For $j = \frac{1}{2}$ we have constructions with components b_r (r are half-integers) to be anticommuting:

$$F_{j=\frac{1}{2}}(z) = \sum_{r=\frac{1}{2}} b_r z^r + \sum_{r=\frac{1}{2}} b_{-r} z^{-r}; \quad r = \frac{1}{2}; \frac{3}{2}; \frac{5}{2}... \quad (13)$$

$$\{b_r, b_s\} = \delta_{r,-s}; \quad b_r^\dagger = b_{-r}; \quad (14)$$

$$L_n = \frac{1}{2}\sum_r (-\frac{n}{2}+r) b_{n-r} b_r; \quad n \neq 0; \quad L_0 = \sum_{r>0} r b_{-r} b_r; \quad L_n^\dagger = L_{-n} \tag{15}$$

Of course we are able to obtain boson representations for arbitrary conformal spin as composite ones from j=0 components. They are exponential vertex operators of Veneziano type [2]:

$$F_j(z) = \exp\left(-k \sum_{n>0} a_{-n} \frac{z^{-n}}{n}\right) \exp\left(ikx_0 + kp \ln z\right) \exp\left(k \sum_{n>0} a_n \frac{z^n}{n}\right); \tag{16}$$

Here

$$[p, x_0] = \frac{1}{i}; \quad j = \frac{k^2}{2} \tag{17}$$

Now we shall give a spinor consruction for $j = \frac{1}{4}$. We use anticommuting spinor components ψ_r (r are multiples of quarters):

$$\left\{\tilde{\psi}_{\alpha,r}, \psi_{\beta,s}\right\} = \delta_{\alpha,\beta}\delta_{r,-s}; \quad \psi^\dagger_{\alpha,r} = \psi_{\alpha,-r}; \quad r = \pm\frac{1}{4}; \pm\frac{3}{4}; \pm\frac{5}{4}... \tag{18}$$

$$\tilde{\psi} = \psi T_0; \qquad (T_0)_{\alpha\beta} = (T_0)_{\beta\alpha} \tag{19}$$

Let us build currents $J(z)$ from these spinor components ψ_r and then we introduce the Sugawara-like operators for L_n (compare [3]):

$$J(z) = \sum_{n=1} J_n z^n + J_0 + \sum_{n=1} J_{-n} z^{-n}; \quad J_n = \sum_r \tilde{\psi}_{n-r} \Gamma \psi_r; \tag{20}$$

$$J_0 = \sum_r \tilde{\psi}_{-r} \tilde{\Gamma} \psi_r; \quad \tilde{\Gamma} = \frac{1}{\sqrt{\rho}} \Gamma \tag{21}$$

Here

$$J_{-n} = J_n^\dagger \quad (T_0\Gamma)_{\alpha\beta} = -(T_0\Gamma)_{\alpha\beta} \quad \Gamma^2 = \rho I; \quad Tr\Gamma^2 = 1; \quad Tr\Gamma = 0$$

$$L_n = \frac{1}{4}\sum_m J_{n-m} J_m; \quad n \neq 0; \quad L_0 = \frac{1}{4} J_0^2 + \frac{1}{2}\sum_{m>0} J_{-m} J_m; \quad L_n^\dagger = L_{-n} \tag{22}$$

J_n satisfy commutation relations:

$$[J_n, J_m] = 2n\delta_{n,-m}; \quad n > 0 \tag{23}$$

L_n satisfy the extended Virasoro algebra (5).

J(z) has the conformal spin j to be 1 in relation to L_n (22). Now we can obtain the spinor representation $\Psi(z)$ for $j = \frac{1}{4}$ in relation to L_n (22) acting on vacuum state $< 0|$.
Here $\psi_{\alpha,r}|0 >=< 0|\psi_{\alpha,-r} = 0$; $r > 0$

$$< 0|\Psi(z) =< 0| \sum_r \Psi_r z^r \tag{24}$$

We take

$$< 0|\Psi_{\frac{1}{4}} =< 0|\psi_{\frac{1}{4}} \tag{25}$$

$$< 0|\Psi_{\frac{1}{4}} L_0 =< 0|\psi_{\frac{1}{4}}(\frac{1}{4}J_0^2) = \frac{1}{4} < 0|\psi_{\frac{1}{4}} \tag{26}$$

Other components of $\Psi(z)$ we are able to obtain using

$$[L_1, \Psi_r] = (-\frac{3}{4} - r)\Psi_{r+1} \tag{27}$$

So in correspondence with (10),(27) we have

$$< 0|\Psi_{\frac{5}{4}} = - < 0| \left[L_1, \psi_{\frac{1}{4}}\right] = \frac{1}{2} < 0|(\tilde{\Gamma}\psi_{\frac{1}{4}})J_1 \tag{28}$$

Similary we have

$$< 0|\Psi_{\frac{9}{4}} = -\frac{1}{2} < 0| \left[L_1, \Psi_{\frac{5}{4}}\right] =< 0|(\frac{1}{4}(\tilde{\Gamma}\psi_{\frac{1}{4}})J_2 + \frac{1}{8}\psi_{\frac{1}{4}}J_1^2) \tag{29}$$

and

$$< 0|\Psi_{\frac{13}{4}} = -\frac{1}{3} < 0| \left[L_1, \Psi_{\frac{9}{4}}\right] =< 0|(\frac{1}{6}(\tilde{\Gamma}\psi_{\frac{1}{4}})J_3 + \frac{1}{8}\psi_{\frac{1}{4}}J_1J_2 + \frac{1}{48}(\tilde{\Gamma}\psi_{\frac{1}{4}})J_1^3) \tag{30}$$

and finally we have

$$< 0|\Psi(z) =< 0|z^{\frac{1}{4}}((\tilde{\Gamma}\psi_{\frac{1}{4}}) \sinh(\sum_{n>0} J_n \frac{z^n}{2n}) + \psi_{\frac{1}{4}} \cosh(\sum_{n>0} J_n \frac{z^n}{2n})) \tag{31}$$

It is easy to find the correct correlation function:

$$< 0|\Psi(z)\Psi^\dagger(1)|0 >= \frac{z^{\frac{1}{4}}}{\rho\sqrt{(1-z)}} \tag{32}$$

The required construction of super Virasoro algebra appears due to introduction of an additional field $\Phi_{j=\frac{1}{2}}(z)$ and corresponding supergenerators:

$$G_r = \frac{1}{\sqrt{2}} \sum_n J_n \Phi_{r-n} \tag{33}$$

$$\Phi_{j=\frac{1}{2}}(z) = \sum_{r=\frac{1}{2}} \Phi_r z^r + \sum_{r=\frac{1}{2}} \Phi_{-r} z^{-r}; \quad r = \frac{1}{2}; \frac{3}{2}; \frac{5}{2}... \tag{34}$$

$$\{\Phi_r, \Phi_s\} = \delta_{r,-s}; \quad \Phi_r^\dagger = \Phi_{-r}; \tag{35}$$

So we obtain the necessary super Virasoro algebra:

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c'}{3}(r^2 - 1/4)\delta_{r,-s} \tag{36}$$

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n,-m} \tag{37}$$

$$[L_n, G_r] = (n/2 - r)G_{n+r} \tag{38}$$

$$L_n = \frac{1}{4} \sum_m J_{n-m} J_m) + \frac{1}{2} \sum_r (-\frac{n}{2} + r)\Phi_{n-r}\Phi_r; \quad n \neq 0 \tag{39}$$

$$L_0 = \frac{1}{4} J_0^2 + \frac{1}{2} \sum_{m>0} J_{-m} J_m + \sum_{r>0} r \Phi_{-r} \Phi_r; \quad L_n^\dagger = L_{-n} \tag{40}$$

Generalization of this formalism for $j = \frac{1}{k}$ with any integer $k > 2$ is evident.

References

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